Adomian decomposition method with Chebyshev polynomials

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Abstract

In this paper an efficient modification of the Adomian decomposition method is presented by using Chebyshev polynomials. The proposed method can be applied to linear and non-linear models. The scheme is tested for some examples and the obtained results demonstrate reliability and efficiency of the proposed method.

Keywords: Adomian decomposition method; Chebyshev polynomials; Ordinary differential equations

1. Introduction

The Adomian decomposition method and its modifications [9,12,14] have efficiently used to solve the ordinary differential equations. It is the purpose of this paper to introduce a new reliable modification of Adomian decomposition method. For this reason, at the beginning of implementation of Adomian method, Chebyshev orthogonal polynomials are used to expand functions. In addition, the proposed modified Adomian decomposition method is

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numerically performed through Maple programming. The obtained results show the advantage using the proposed modified Adomian decomposition method.

It is well known that the eigenfunctions of certain singular Sturm–Liouville problems allow the approximation of functions \( C^\infty[a,b] \) where truncation error approaches zero faster than any negative power of the number of basic functions used in the approximation, as that number (order of truncation \( M \)) tends to infinity [4]. This phenomenon is usually referred to as “spectral accuracy” [5]. The accuracy of derivatives obtained by direct, term-by-term differentiation of such truncated expansion naturally deteriorates [4], but for low-order derivatives and sufficiently high order truncations this deterioration is negligible, compared to the restrictions in accuracy introduced by typical difference approximations (for more details, refer to [2,7]). Throughout, we are using first kind orthogonal Chebyshev polynomials \( f_{T_k}^{g+1} \), which are eigenfunctions of singular Sturm–Liouville problem

\[
\left( \sqrt{1-x^2} T'(x) \right)' + \frac{k^2}{\sqrt{1-x^2}} T_k(x) = 0.
\]

2. Modified Adomian decomposition method

Here, the review of the standard Adomian decomposition method is presented. For this reason, consider the differential equation,

\[
Lu + Ru + Nu = g(x),
\]

where \( N \) is a non-linear operator, \( L \) is the highest-order derivative which is assumed to be invertible, \( R \) is a linear differential operator of less order than \( L \) and \( g \) is the source term.

The method is based on applying the operator \( L^{-1} \) formally to the expression

\[
Lu = g - Ru - Nu,
\]

so, by using the given conditions we obtain

\[
u = f - L^{-1}(Ru) - L^{-1}(Nu),\]

where the function \( f \) represents the terms arising from integrating the source term \( g \), \( L^{-1}g \), and from using the given conditions, \( \Phi(x) \), all are assumed to be prescribed.

We write \( Nu = \sum_{n=0}^{+\infty} A_n \) and \( u = \sum_{n=0}^{+\infty} u_n \), where the components of \( A_n \) are the so-called Adomian polynomials, for each \( i, A_i \) depends on \( u_0, u_1, \ldots, u_i \) only. Now, by considering (3), we define
\[
\begin{aligned}
\begin{cases}
  u_0 &= L^{-1}(g) + \Phi(x), \\
  u_1 &= -L^{-1}(Ru_0) - L^{-1}(Nu_0), \\
  u_2 &= -L^{-1}(Ru_1) - L^{-1}(Nu_1)
\end{cases}
\end{aligned}
\]  
\tag{4a}

and in general
\[
u_{k+1} = -L^{-1}(Ru_k) - L^{-1}(Nu_k), \quad k \geq 0.
\]  
\tag{4b}

If the series converges in a suitable way, then we can see that
\[
u = \lim_{M \to +\infty} \Psi_M(x),
\]
where \( \Psi_M(x) = \sum_{i=0}^{M} u_i \). Now, we require on expression for the \( A_i \), Specific algorithms were seen in \([8,10,11]\) to formulate Adomian polynomials. The following algorithm:
\[
\begin{align*}
  A_0 &= F(u_0), \\
  A_1 &= u_1 F'(u_0), \\
  A_2 &= u_2 F'(u_0) + \frac{1}{2} u_1^2 F''(u_0), \\
  A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0), \\
  \vdots
\end{align*}
\]
can be used to construct Adomian polynomials, when \( F(u) = Nu \).

The theoretical treatment of the convergence of Adomian decomposition method has been considered in \([1,3,6,13]\).

To perform the Adomian decomposition method, in general, for an arbitrary natural number \( v \), \( g(x) \) is expressed in Taylor series,
\[
g(x) \approx \sum_{i=0}^{v} g_i(x). \tag{5}
\]

In this paper, we suggest that \( g(x) \) be expressed in Chebyshev series,
\[
g(x) \approx \sum_{i=0}^{v} a_i T_i(x), \tag{6}
\]
where \( T_i(x) \) is the first kind of orthogonal Chebyshev polynomial,
\[
\begin{align*}
  T_0(x) &= 1, \\
  T_1(x) &= x, \\
  T_2(x) &= 2x^2 - 1, \\
  T_3(x) &= 4x^3 - 3x
\end{align*}
\]
and in general,
\[ T_{k+1} = 2xT_k - T_{k-1}, \quad k \geq 1. \]

Now, using (4) and (6) we have
\[
\begin{align*}
\begin{cases}
u_0 & = L^{-1}(a_0 T_0(x)) + a_1 T_1(x) + a_2 T_2(x) + \cdots + a_v T_v(x) + \Phi(x), \\
u_1 & = -L^{-1}(Ru_0) - L^{-1}(Nu_0), \\
u_2 & = -L^{-1}(Ru_1) - L^{-1}(Nu_1), \\
& \vdots
\end{cases}
\end{align*}
\]

(7)

We will show that the above obtained approximate solution, \( \Psi_M(x) = \sum_{i=0}^{M} u_i \), is more accurate and efficient than the obtained approximate solution by (4) and (5).

In addition, according to [14], we can put,
\[
\begin{align*}
\begin{cases}
u_0 & = L^{-1}(a_0 T_0(x)) + \Phi(x), \\
u_1 & = L^{-1}(a_1 T_1(x)) + L^{-1}(Ru_0) - L^{-1}(Nu_0), \\
u_2 & = L^{-1}(a_2 T_2(x)) - L^{-1}(Ru_1) - L^{-1}(Nu_1), \\
& \vdots
\end{cases}
\end{align*}
\]

(8)

or by converting (6) into standard form, we have
\[
g(x) \approx b_0 + b_1 x + b_2 x^2 + \cdots + b_v x^v,
\]

\[
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
\vdots \\
b_v
\end{bmatrix} = \begin{bmatrix}
1 & 0 & -1 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & -3 & 0 & 5 & \cdots \\
0 & 0 & 2 & 0 & -8 & 0 & \cdots \\
0 & 0 & 0 & 4 & 0 & -20 & \cdots \\
0 & 0 & 0 & 0 & 8 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_v
\end{bmatrix}
\]

and
\[
\begin{align*}
\begin{cases}
u_0 & = L^{-1}(b_0) + \Phi(x), \\
u_1 & = L^{-1}(b_1) + L^{-1}(Ru_0) - L^{-1}(Nu_0), \\
u_2 & = L^{-1}(b_2) - L^{-1}(Ru_1) - L^{-1}(Nu_1), \\
& \vdots
\end{cases}
\end{align*}
\]

(9)

The obtained approximate solution, \( \Psi_M(x) = \sum_{i=0}^{M} u_i \), by (7), (8) or (9), has similar behavior compared to the Chebyshev expansion of the exact solution, \( u(x) \).
3. Test problems

In this section, two initial ordinary differential equations are considered and these problems are solved by Adomian decomposition method (4) and (5), \( u_T(x) \), and proposed Adomian decomposition method (7), \( u_C(x) \). The algorithms are performed by Maple 8 with 10 digits precision.

**Example 1.** Consider for \( 0 \leq x \leq 1 \)

\[
\begin{align*}
\frac{d^2u}{dx^2} + xu' + x^2 u^3 &= (2 + 6x^2)e^x + x^2 e^{3x^2}, \\
u(0) &= 1, \quad u'(0) = 0,
\end{align*}
\tag{10a}
\]

with the exact solution \( u(x) = e^{x^2} \). According to 1 we have,

\[
Lu + Ru + Nu = g(x),
\]

where \( L = \frac{d}{dx} \), \( R = x \frac{d}{dx} \), \( Nu = x^2 u^3 \) and \( g(x) = (2 + 6x^2)e^x + x^2 e^{3x^2} \). In addition, \( F(u) = Nu = x^2 u^3 \), So, the Adomian polynomials are,

\[
\begin{align*}
A_0 &= x^2 u_0^3, \\
A_1 &= x^2(3u_0^3 u_1), \\
A_2 &= x^2(3u_0^2 u_2 + 3u_0 u_1^2), \\
A_3 &= x^2(3u_0^2 u_3 + 6u_0 u_2 u_1 + u_1^3), \\
&\vdots
\end{align*}
\]

Now let \( v = M = 6 \), since \( L^{-1} = \int_0^x \int_0^x (\cdot) \, dx \, dx \) and the Taylor series of \( g(x) \) is

\[
g(x) \approx 2 + 9x^2 + 10x^4 + \frac{47}{6} x^6 + O(x^7).
\]

So, by (4), we have,

\[
\begin{align*}
u_0 &= L^{-1} \left( 2 + 9x^2 + 10x^4 + \frac{47}{6} x^6 \right) + u(0) + u'(0)x = 1 + x^2 + \frac{3}{4} x^4 + \frac{1}{3} x^6 + \frac{47}{336} x^8, \\
u_1 &= -L^{-1} \left( x \frac{d}{dx} u_0 \right) - L^{-1}(A_0) = -\frac{1}{4} x^4 - \frac{1}{5} x^6 - \frac{29}{224} x^8 + \cdots, \\
u_2 &= -L^{-1} \left( x \frac{d}{dx} u_1 \right) - L^{-1}(A_1) = \frac{1}{30} x^6 + \frac{39}{1120} x^8 + \frac{439}{12,600} x^{10} + \cdots, \\
u_3 &= -L^{-1} \left( x \frac{d}{dx} u_2 \right) - L^{-1}(A_2) = \frac{1}{280} x^8 - \frac{53}{12,600} x^{10} + \cdots, \\
u_4 &= -L^{-1} \left( x \frac{d}{dx} u_3 \right) - L^{-1}(A_3) = \frac{4}{12,600} x^{10} + \cdots, \\
&\vdots
\end{align*}
\]
and we obtain

$$u_T(x) = S_6(x) = \sum_{i=0}^{6} u_i = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} - \frac{29}{540} x^{10} + \cdots.$$ 

The absolute error of $u_T(x)$ is shown in Fig. 1. It must be noted that, even if we put

$$u_T(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24},$$

the maximum norm error will be 0.01, i.e., $\|u_T - u\|_\infty = 0.01$.

Now, we use the Chebyshev expansion for $g(x)$ and the relation (7). In this case we have,

$$g(x) \approx \sum_{i=0}^{6} a_i T_i(2x - 1), \quad 0 \leq x \leq 1,$$

where

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{g(0.5x + 0.5)T_0(x)}{\sqrt{1 - x^2}} \, dx,$$

$$a_i = \frac{2}{\pi} \int_{-1}^{1} \frac{g(0.5x + 0.5)T_i(x)}{\sqrt{1 - x^2}} \, dx, \quad i = 1, 2, \ldots, 6$$

and it implies that,

$$g(x) \approx 2.2023 - 2.2154x + 43.1307x^2 + \cdots + 258.2588x^6,$$

Fig. 1. $|u(x) - u_T(x)|$ for Example 1.
\[ u_0 = L^{-1}(2.2023 - 2.2154x + 43.1307x^2 + \cdots + 258.2588x^6) + u(0) + u'(0)x = 1 + 1.0114x^2 + \cdots + 4.6118x^8, \]

\[ u_c(x) = \sum_{i=0}^{6} u_i = 1 + 1.0114x^2 - 0.3692x^3 + 3.3423x^4 - 9.4415x^5 + 15.6713x^6 - 12.0886x^7 + 2.6988x^8 + \cdots. \]

The absolute error of \( u_c(x) \) is presented in Fig. 2.

**Example 2.** Consider for \( 0 \leq x \leq 1, \)

\[ u'' + uu' = x \sin(2x^2) - 4x^2 \sin(x^2) + 2 \cos(x^2), \]

\[ u(0) = u'(0) = 0, \]

with the exact solution \( u(x) = \sin(x^2) \). Here \( Nu = F(u) = uu' \), so according to [11], the Adomian polynomials are as below,

\[ \begin{aligned}
A_0 &= u_0u_0', \\
A_1 &= u_1u_0' + u_0u_1', \\
A_2 &= u_2u_0' + u_1u_1' + u_0u_2', \\
A_3 &= u_3u_0' + u_2u_1' + u_1u_2' + u_0u_3', \\
&\vdots
\end{aligned} \]

Now, let \( \nu = M = 10 \), if we expand \( g(x) \) by Taylor series (5) and use (4), we have

\[ \text{Fig. 2. } |u(x) - u_c(x)| \text{ for Example 1.} \]
\[ u_T(x) = S_{10}(x) = \sum_{i=0}^{10} u_i = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{13}}{585} + \frac{x^{16}}{9360} + \cdots. \]

Also, by (7), we obtain,

\[ u_C(x) = \sum_{i=0}^{10} u_i = 1.0000x^2 - 0.0005x^4 + 0.0044x^5 + \cdots - 0.1603x^{10}. \]

Figs. 3 and 4, show the absolute error of approximate solutions \( u_T(x) \) and \( u_C(x) \), respectively.

![Graph 3](image1.png)

**Fig. 3.** \( |u(x) - u_T(x)| \) for Example 2.

![Graph 4](image2.png)

**Fig. 4.** \( |u(x) - u_C(x)| \) for Example 2.
The comparison between the results mentioned in Figs. 1–4 shows the power of the proposed method of this paper, for these examples.

References