A modified pseudospectral method for numerical solution of ordinary differential equations systems

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Abstract

In this paper, numerical solution of first-order systems of ordinary differential equations (ODEs) is considered and an error estimation method is proposed. By this method, the efficiency of pseudospectral method to solve the systems is determined and a modified pseudospectral method is presented. Furthermore, with providing some examples, the aforementioned cases are dealt with numerically.

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1. Introduction

The numerical solution of systems of ordinary differential equations has been considered in [6,10,11]. It is a purpose of this paper to introduce a new error estimation method when pseudospectral method is applied. In addition, this proposed error estimation method will be used to present a modified pseudospectral method. Some examples are numerically solved by the proposed methods and the results show the advantage using them.

It is known that the eigenfunctions of certain singular Sturm–Liouville problems allow the approximation of functions in $C^+[a,b]$ where truncation error approaches zero faster than any negative power of the number of basic functions used in the approximation, as that number (order of truncation $N$) tends to infinity [3]. This phenomenon is usually referred to as “spectral accuracy” [8]. The accuracy of derivatives obtained by direct, term-by-term differentiation of such truncated expansion naturally deteriorates [7], but for low-order derivatives and sufficiently high order truncations this deterioration is negligible, compared to the restrictions in accuracy introduced by typical difference approximations (for more details, refer to [1,4]). Throughout, we are using first kind orthogonal Chebyshev polynomials $\{T_k\}_{k=0}^{+\infty}$ which are eigenfunctions of singular Sturm–Liouville problem

$$\left(\sqrt{1-x^2}T'_k(x)\right)' + \frac{k^2}{\sqrt{1-x^2}} T_k(x) = 0.$$
2. Error estimation method

Here, the review of the pseudospectral method is presented. For this reason, consider the linear system of ODEs:

\[ \sum_{j=1}^{n} b_{ij}(t) x_j' + \sum_{j=1}^{n} c_{ij}(t) x_j = f_i(t) \quad i = 1, 2, \ldots, n, \]  
\[ x_i(t_0) + \beta_i x_i(t_f) = \gamma_i \quad i = 1, 2, \ldots, n, \]

where \( b_{ij}, c_{ij} \) and \( f_i \) are continuous functions of \( t \), \( t_0 \leq t \leq t_f \) and \( x_i, \beta_i \) and \( \gamma_i \) are constants. Here, the implementation of pseudospectral method is presented for ODEs system (1), when \( n = 3 \). This discussion can simply be extended to general form (1). For an arbitrary natural number \( v \), we suppose that the approximate solution of ODEs system (1) is as below:

\[ x_1(t) = \sum_{i=0}^{v} a_i T_i(s), \]  
\[ x_2(t) = \sum_{i=0}^{v} a_{i+1} T_i(s), \quad s \in [-1,1], \]  
\[ x_3(t) = \sum_{i=0}^{v} a_{2i+3} T_i(s), \]

where

\[ t = h(s) = \frac{t_f - t_0}{2} s + \frac{t_f + t_0}{2}, \]  
\[ a = (a_0, a_1, \ldots, a_{3v+2})^T \in \mathbb{R}^{3v+3} \text{ and } \{ T_k \}_{k=0}^{v} \text{ is sequence of Chebyshev polynomials of the first kind}. \]

The main purpose is to find \( a = (a_0, a_1, \ldots, a_{3v+2})^T \). Now, by using (3), we rewrite system (1), as below:

\[ \sum_{j=1}^{v} b_{ij}(h(s)) x_j' + \sum_{j=1}^{v} c_{ij}(h(s)) x_j = f_i(h(s)) \quad i = 1, 2, 3, \]  
\[ x_i(-1) + \beta_i x_i(1) = \gamma_i \quad i = 1, 2, 3. \]

Substituting (2) into (4), implies that (for more details refer to [1,2])

\[ \sum_{j=0}^{3v+2} a_j \Phi_j(s) \approx f_i(h(s)), \quad i = 1, 2, 3, \]

and by substituting Chebyshev–Gauss points [5]

\[ s_k = \cos \left( \frac{k\pi}{v} \right), \quad k = 0, 1, \ldots, v-1 \]

into (5), a linear system with \( 3v \) equations and \( 3v + 3 \) unknowns is obtained. To construct the remaining three equations (by attending to boundary conditions (4b)), we put

\[ x_i(-1) + \beta_i x_i(1) = \gamma_i \Rightarrow x_i \sum_{k=0}^{v} a_k T_k(-1) + \beta_i \sum_{k=0}^{v} a_k T_k(1) = \gamma_i \]

\[ \Rightarrow x_i \sum_{k=0}^{v} a_k (-1)^k + \beta_i \sum_{k=0}^{v} a_k = \gamma_i, \quad i = 1, 2, 3 \]

(7)

to obtain three equations. In addition, according to given boundary (or initial) conditions, the collocation points can be chosen by different manners [3] in (6).
Now, suppose that ODEs (4) is solved by pseudospectral method and
\[
\begin{align*}
x_{1i}(t) &= \sum_{i=0}^{r} a_i T_i(s), \\
x_{2i}(t) &= \sum_{i=0}^{r} a_{i+i+1} T_i(s), \\
x_{3i}(t) &= \sum_{i=0}^{r} a_{2i+i+2} T_i(s),
\end{align*}
\]
are obtained. So, we have
\[
d\frac{ds}{dt} \sum_{j=1}^{n} b_{ij}(h(s)) x'_j + \sum_{j=1}^{n} c_{ij}(h(s)) x_j - f_i(h(s)), \quad i = 1, 2, 3
\]
with boundary conditions
\[
x_0 x_i(-1) + \beta_i x_i(1) = \gamma_{1i}, \quad i = 1, 2, 3.
\]
We define \( E_v \in R^3 \), as below:
\[
E_v = \frac{ds}{dt} \sum_{j=1}^{n} b_{ij}(h(s)) x'_j + \sum_{j=1}^{n} c_{ij}(h(s)) x_j - f_i(h(s)), \quad i = 1, 2, 3,
\]
and by considering (4) and (11), we have
\[
\frac{ds}{dt} \sum_{j=1}^{n} b_{ij}(h(s)) e'_j + \sum_{j=1}^{n} c_{ij}(h(s)) e_j = -E_v(s), \quad i = 1, 2, 3,
\]
\[
x_0 e_v(-1) + \beta_i e_v(1) = 0, \quad i = 1, 2, 3,
\]
where \( e_v = x_i - x_{iv} \).

Here, we call \( e_v(t) \) as error function and if we put
\[
\begin{align*}
\tilde{e}_{1i}(t) &= \sum_{i=0}^{r} \tilde{a}_i T_i(s), \\
\tilde{e}_{2i}(t) &= \sum_{i=0}^{r} \tilde{a}_{i+i+1} T_i(s), \\
\tilde{e}_{3i}(t) &= \sum_{i=0}^{r} \tilde{a}_{2i+i+2} T_i(s).
\end{align*}
\]
Eq. (5) implies that
\[
\sum_{j=0}^{3r+2} \tilde{a}_j \Phi_{ij}(s) \approx -E_v(h(s)), \quad i = 1, 2, 3.
\]
Now, we put
\[
\begin{align*}
\sum_{j=0}^{3r+2} \tilde{a}_j \Phi_{ij}(s) &= -E_v(h(s)), \quad i = 1, 2, 3, \\
x_0 e_v(-1) + \beta_i e_v(1) &= 0, \quad i = 1, 2, 3.
\end{align*}
\]
By substituting the collocation points (6) into (15) and with solving the obtained linear system, the \( \tilde{e}_v(t) \) will be determined. But, in this manner \( \tilde{e}_v(t) \) cannot appropriately estimate the error function \( e_v(t) \), because we obtained \( x_v(t), i = 1, 2, 3 \), by substituting the collocation points (6) and again, we used of these points to compute \( E_v(t) \). So the norm of obtained \( E_v(t) \) will be very small and \( \tilde{a} = (\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_{3r+2})' \in R^{3r+3} \), and consequently \( \tilde{e}_v(t) \), will be approximately obtained as vanish (the similar result obtain when we use of the proposed algo-
rithm mentioned in [9]). To remove this difficulty, we note to this fact, relation (14) is defined for every points belong to \([-1, 1]\), so, for computing, \(e_i(t)\), we can use of other appropriate points. For this reason, consider the \(\nu\) zero points of \(T_{\nu}(s)\), i.e.

\[ s_k = \cos \left( \frac{(2k + 1)\pi}{2\nu} \right), \quad k = 0, 1, \ldots, \nu - 1. \]

(16)

Here, every points of (16) is exactly between two points of (6) [3,5]. Now by substituting points (16) into (15a) and with solving the obtained linear system, the obtained \(e_i(t)\) will be appropriately estimated the error function \(e_i(t)\). Here, we conclude that obtained approximate solution is not well if the \(e_i(t)\) has a large norm value and in this case, if we introduce

\[ x_i = x_i + \bar{e}_i, \]

(17)

the obtained solutions are more accurate than \(x_i\). In Section 3, some examples are solved by pseudospectral and modified pseudospectral (17) methods and the presented discusses are numerically illustrated.

3. Numerical results

In this section, two examples are numerically solved by pseudospectral method and the error estimation technique (mentioned in Section 2) is applied for them. The obtained results show that \(\bar{e}_i\) can appropriately estimate the exact error \(|x - x_i|\), and the obtained approximated solution \(\bar{x}_i\) (obtained by modified pseudospectral method) is more accurate than \(x_i\) (obtained by pseudospectral method). Furthermore, the presented algorithms in Section 2 are performed using Maple 8 with 20 digits precision.

Example 1 [10]. Consider the system of first-order ordinary differential equations

\[
\begin{cases}
t^2x_1' + \frac{2}{x}x_1 - 2x_2 = 0, & 1 \leq t \leq 2 \\
\frac{2}{t}x_1 + \frac{1}{2}x_2 + tx_2' = 0, & 1 \leq t \leq 2
\end{cases}
\]

with the initial conditions \(x_1(1) = 1\) and \(x_2(1) = 0.5\). The analytical solution of this problem is

![Fig. 1. Absolute errors \(|x_1 - x_i|\), \(\bar{e}_i\), and \(|\bar{e}_i(t)|\).](image-url)
\[ x_1(t) = t^{-\frac{1}{2}}(1 + 2 \ln(t)), \]
\[ x_2(t) = t^{\frac{1}{2}} \left( \frac{1}{2} - \ln(t) \right). \]

In Fig. 1, we report the absolute exact error \(|x_1 - x_1v|\) and absolute estimated error \(|\tilde{e}_{1v}(t)|\) such that \(x_1v\) is obtained by applying pseudospectral method (with \(v = 8\)) for this problem. Here, the exact and the estimated errors are approximately equal and it shows that the proposed estimation error method can appropriately estimate the exact error for this problem. Also, by considering (17), the approximate solution \(\tilde{x}_v\), is obtained and Table 1 displays the maximum norm errors \(\|X - X_v\|_\infty\) and \(\|X - \tilde{X}_v\|_\infty\). These values are approximately obtained through their graphs.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Maximum norm error for Example 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v)</td>
<td>Pseudospectral method</td>
</tr>
<tr>
<td>(|x_1 - x_1v|_\infty)</td>
<td>(|x_2 - x_2v|_\infty)</td>
</tr>
<tr>
<td>8</td>
<td>1.8(–4)</td>
</tr>
<tr>
<td>12</td>
<td>5.2(–7)</td>
</tr>
<tr>
<td>16</td>
<td>1.5(–9)</td>
</tr>
</tbody>
</table>

![Graph showing the absolute errors for x1, x2, and the estimated errors.](image)

Fig. 2. Absolute errors \(|x_1 - x_{1v}|\), ‘—’, and \(|\tilde{e}_{1v}(t)|\), ‘- - -’.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Maximum norm error for Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v)</td>
<td>Pseudospectral method</td>
</tr>
<tr>
<td>(|x_1 - x_1v|_\infty)</td>
<td>(|x_4 - x_{4v}|_\infty)</td>
</tr>
<tr>
<td>9</td>
<td>1.2(–6)</td>
</tr>
<tr>
<td>15</td>
<td>3.2(–15)</td>
</tr>
<tr>
<td>21</td>
<td>2.0(–15)</td>
</tr>
</tbody>
</table>
Example 2. Consider for \(-1 \leq t \leq 1\)

\[
AX' + BX = g(t),
\]

\[
A = \begin{bmatrix}
  x + 2 & x + 1 & 0 & x^2 \\
  x & x & x^2 + x & x + 1 \\
  0 & 0 & \sin(x) + 1 & 0 \\
  0 & \cos(x) + x & 0 & x + 1
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
  |x| + 1 & x + 2 & x^2 + 1 & |x| + 1 \\
  x + 2 & |x| + 1 & |x| + 1 & x^2 + 1 \\
  |x| + 1 & \cos(x) + 2 & |x| + 1 & x + 2 \\
  |x| + 1 & 1 & \sin(x) + 2 & |x| + 1
\end{bmatrix}
\]

with boundary conditions

\[x_i(-1) + x_i(1) = z_i, \quad i = 1, 2, 3, 4,\]

where \(g(t)\) and \(z_i\) are chosen such that the exact solutions are

\[x_1(t) = e^t, \quad x_2(t) = \sin(t), \quad x_3(t) = e^{-t}, \quad x_4(t) = \cos(t).\]

This problem is solved by pseudospectral method with \(v = 9\). Fig. 2 shows the absolute error of exact error \(|x_4 - x_{4i}|\) and absolute estimated error \(|\tilde{e}_{4i}(t)|\). Here, exact and estimated errors are approximately equal. Furthermore, this example is solved by pseudo- and modified-pseudospectral methods, Table 2.

The results show the efficiency of error estimation and modified pseudospectral methods for above examples.

References


